

# NOTE ON TWISTED ELLIPTIC GENUS OF $K3$ SURFACE

TOHRU EGUCHI AND KAZUHIRO HIKAMI

**ABSTRACT.** We discuss the possibility of Mathieu group  $M_{24}$  acting as symmetry group on the  $K3$  elliptic genus as proposed recently by Ooguri, Tachikawa and one of the present authors. One way of testing this proposal is to derive the twisted elliptic genera for all conjugacy classes of  $M_{24}$  so that we can determine the unique decomposition of expansion coefficients of  $K3$  elliptic genus into irreducible representations of  $M_{24}$ . In this paper we obtain all the hitherto unknown twisted elliptic genera and find a strong evidence of Mathieu moonshine.

## 1. INTRODUCTION

Let us consider the string theory compactified on the  $K3$  surface. It is well-known that string theory on  $K3$  has the symmetry of  $\mathcal{N} = 4$  superconformal algebra. In [6, 10] the elliptic genus of  $K3$  was expanded in terms of irreducible representations of  $\mathcal{N} = 4$  superconformal algebra

$$\begin{aligned} Z_{K3}(z; \tau) &= 8 \left[ \left( \frac{\theta_{10}(z; \tau)}{\theta_{10}(0; \tau)} \right)^2 + \left( \frac{\theta_{00}(z; \tau)}{\theta_{00}(0; \tau)} \right)^2 + \left( \frac{\theta_{01}(z; \tau)}{\theta_{01}(0; \tau)} \right)^2 \right] \\ &= 20 \operatorname{ch}_{h=\frac{1}{4}, \ell=0}^{\tilde{R}}(z; \tau) - 2 \operatorname{ch}_{h=\frac{1}{4}, \ell=\frac{1}{2}}^{\tilde{R}}(z; \tau) + \sum_{n=1}^{\infty} A(n) \operatorname{ch}_{h=n+\frac{1}{4}, \ell=\frac{1}{2}}^{\tilde{R}}(z; \tau). \end{aligned} \quad (1.1)$$

In the RHS the first two terms denote characters of short (BPS) representations of  $\mathcal{N} = 4$  algebra (with isospin  $\ell = 0, 1/2$ ), and an infinite series denotes the sum over long (non-BPS) representations. A new discovery was made in [9]: expansion coefficients  $A(n)$  agree with the dimensions of irreducible or reducible dimensions of representations of the Mathieu group  $M_{24}$ ,

$n$	1	2	3	4	5	6	7	8	$\dots$
$A(n)$	$2 \times 45$	$2 \times 231$	$2 \times 770$	$2 \times 2277$	$2 \times 5796$	$2 \times 13915$	$2 \times 30843$	$2 \times 65550$	$\dots$

(1.2)

In fact  $M_{24}$  has 26 representations with dimensions

$$\{1, 23, 252, 253, 1771, 3520, 45, \overline{45}, 990, \overline{990}, 1035, \overline{1035}, 1035', 231, \overline{231}, 770, \overline{770}, 483, 1265, 2024, 2277, 3312, 5313, 5796, 5544, 10395\}.$$

Here the pairs of 45, 990, 1035, 231, 770-dimensional representations are complex conjugate of each other. At the level  $n = 6$  we have a decomposition into irreducible representations  $13915 = 3520 + 10395$  and at  $n = 7$   $30843 = 10395 + 5796 + 5544 + 5313 + 2024 + 1171$  etc.

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Explicitly characters of short and long representations (in  $\tilde{R}$  sector) are given by [11]

$$\text{ch}_{h=\frac{1}{4}, \ell=0}^{\tilde{R}}(z; \tau) = \frac{[\theta_{11}(z; \tau)]^2}{[\eta(\tau)]^3} \mu(z; \tau), \quad (1.3)$$

$$\text{ch}_{h=\frac{1}{4}, \ell=\frac{1}{2}}^{\tilde{R}}(z; \tau) + 2 \text{ch}_{h=\frac{1}{4}, \ell=0}^{\tilde{R}}(z; \tau) = q^{-\frac{1}{8}} \frac{[\theta_{11}(z; \tau)]^2}{[\eta(\tau)]^3}, \quad (1.4)$$

$$\mu(z; \tau) = \frac{i e^{\pi i z}}{\theta_{11}(z; \tau)} \sum_{n \in \mathbb{Z}} (-1)^n \frac{q^{\frac{1}{2}n(n+1)} e^{2\pi i n z}}{1 - q^n e^{2\pi i z}}, \quad (1.5)$$

and

$$\text{ch}_{h, \ell=\frac{1}{2}}^{\tilde{R}}(z; \tau) = q^{h-\frac{3}{8}} \frac{[\theta_{11}(z; \tau)]^2}{[\eta(\tau)]^3}. \quad (1.6)$$

It is known that the above series  $\mu(z; \tau)$  is a typical example of a Mock theta function [25]. Expansion coefficients  $A(n)$  are given in terms of  $\mu$  at half-periods

$$-q^{\frac{1}{8}} \Sigma(\tau) \equiv -2 + \sum_{n=1}^{\infty} A(n) q^n = 8 \sum_{w \in \{\frac{1}{2}, \frac{1+\tau}{2}, \frac{\tau}{2}\}} \mu(w; \tau). \quad (1.7)$$

Prefactor  $q^{1/8}$  above is for convenience. As studied in detail in [6],  $\Sigma(\tau)$  is a mock theta function whose shadow [24] is  $[\eta(\tau)]^3$ .

Now consider a graded vector space

$$\sum_{n=1}^{\infty} V(n) q^n$$

where the space  $V(n)$  has a dimension  $\dim V(n) = A(n)$ . Since the dimension of the representation space is given by the trace of the identity element, we may rewrite the sum (1.7) as

$$-q^{\frac{1}{8}} \Sigma(\tau) = -2 + \sum_{n=1}^{\infty} \text{Tr}_{V(n)} 1 \cdot q^n. \quad (1.8)$$

Twisted elliptic genus is defined instead by considering an arbitrary group element  $g$

$$-q^{\frac{1}{8}} \Sigma_g(\tau) = -2 + \sum_{n=1}^{\infty} A_g(n) q^n, \quad (1.9)$$

where

$$A_g(n) = \text{Tr}_{V(n)} g. \quad (1.10)$$

Since the trace of  $g$  depends only on its conjugacy class, there exists a twisted elliptic genus  $Z_g(z; \tau)$  corresponding to each conjugacy class. As a generalization of (1.1), we define the twisted elliptic genus by the decomposition

$$\begin{aligned} Z_g(z; \tau) &= (\chi_g - 4) \text{ch}_{h=\frac{1}{4}, \ell=0}^{\tilde{R}}(z; \tau) - 2 \text{ch}_{h=\frac{1}{4}, \ell=\frac{1}{2}}^{\tilde{R}}(z; \tau) + \sum_{n=1}^{\infty} A_g(n) \text{ch}_{h=n+\frac{1}{4}, \ell=\frac{1}{2}}^{\tilde{R}}(z; \tau) \\ &= \chi_g \text{ch}_{h=\frac{1}{4}, \ell=0}^{\tilde{R}}(z; \tau) - \frac{[\theta_{11}(z; \tau)]^2}{[\eta(\tau)]^3} \Sigma_g(\tau), \end{aligned} \quad (1.11)$$

where  $\chi_g \in \mathbb{Z}$  is the Witten index  $Z_g(z=0; \tau)$ . The  $q$ -series  $\Sigma_g(\tau)$  is thus the analogue of the McKay–Thompson series of the monstrous moonshine [4].

The Mathieu group  $M_{24}$  has the following 26 conjugacy classes: (we use the ATLAS naming of the conjugacy classes [3]. See Table 1)

$$\text{type I : } 1A, 2A, 3A, 5A, 4B, 7A, 7B, 8A, 6A, 11A, 15A, 15B, 14A, 14B, 23A, 23B, \quad (1.12)$$

$$\text{type II : } 12B, 6B, 4C, 3B, 2B, 10A, 21A, 21B, 4A, 12A. \quad (1.13)$$

Elements of the conjugacy classes of the type I (1.12) fix 1 element out of 24 and these classes may be considered as conjugacy classes of the subgroup  $M_{23}$ . See the cycle representation of these classes in Table 1. On the other hand, classes of the type II (1.13) do not have a fixed element and are regarded as intrinsic elements of  $M_{24}$ . It turns out that the twisted elliptic genera of these two types of conjugacy classes have a qualitatively different behavior. Especially the Witten index  $\chi_g$  in (1.11) vanishes if and only if  $g \in$  type II;

$$\begin{array}{c|cccccccccccccc} g & 1A & 2A & 3A & 5A & 4B & 7A & 8A & 6A & 11A & 15A & 14A & 23A & \text{others} \\ \hline \chi_g & 24 & 8 & 6 & 4 & 4 & 3 & 2 & 2 & 2 & 1 & 1 & 1 & 0 \end{array} \quad (1.14)$$

It is observed [2, 13] that  $\chi_g$  is related to the first and the second rows of the character table of  $M_{24}$  in Table 2;  $\chi_{1A} = 1 + 23$ ,  $\chi_{2A} = 1 + 7$ , and so on. In the character decomposition (1.11),  $A_g(n)$  is the Fourier coefficient of mock theta functions if  $g \in$  type I and of modular form if  $g \in$  type II. Note that twisted elliptic genera for the pairs,  $(7A, 7B)$ ,  $(15A, 15B)$ ,  $(14A, 14B)$ ,  $(23A, 23B)$ ,  $(21A, 21B)$ , are equal to each other.

Twisted elliptic genera of the conjugacy classes of type I have already been obtained in the literature [2, 13]. On the other hand, twisted elliptic genera of type II are yet largely unknown. In this paper we obtain all the twisted elliptic genera of type II and then use the character formula of the Mathieu group to derive the coefficients of the decomposition of  $K3$  elliptic genus into a sum of irreducible representations of  $M_{24}$ . We have checked that we always obtain the positive integral coefficients in the decomposition up to  $q^{600}$ . We thus provide a very strong support for the Mathieu moonshine conjecture.<sup>1</sup>

## 2. TWISTED ELLIPTIC GENUS OF THE SET TYPE I

Twisted elliptic genera of the type I has been obtained previously [2, 5, 13]. Type I genera for basic classes,  $pA$  ( $p = 2, 3, 5, 7$ ) were discussed by A.Sen and his collaborators [5, 16] (also [15]) in connection with the counting problem of  $\frac{1}{4}$  BPS monopoles and dyons.

We first introduce the standard notation in the theory of Jacobi forms [12]

$$\phi_{0,1}(z; \tau) = 4 \left[ \left( \frac{\theta_{10}(z; \tau)}{\theta_{10}(0; \tau)} \right)^2 + \left( \frac{\theta_{00}(z; \tau)}{\theta_{00}(0; \tau)} \right)^2 + \left( \frac{\theta_{01}(z; \tau)}{\theta_{01}(0; \tau)} \right)^2 \right], \quad (2.1)$$

and

$$\phi_{-2,1}(z; \tau) = -\frac{[\theta_{11}(z; \tau)]^2}{[\eta(\tau)]^6}. \quad (2.2)$$

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<sup>1</sup>Very recently a preprint [14] by M.Gaberdiel, S.Hohenegger and R. Volpato has appeared which has a substantial overlap with the present paper.

Here the Jacobi theta functions are defined in Appendix A.  $\phi_{M,N}$  denotes a Jacobi form with weight  $M$  and index  $N$ . We also use the Eisenstein series

$$\phi_2^{(N)}(\tau) = \frac{24}{N-1} q \partial_q \log \left( \frac{\eta(N\tau)}{\eta(\tau)} \right) = 1 + \frac{24}{N-1} \sum_{k=1}^{\infty} \sigma_1(k) (q^k - N q^{Nk}). \quad (2.3)$$

The elliptic genus for  $K3$  is given by [10, 17]

$$Z_{K3}(z; \tau) = Z_{1A}(z; \tau) = 2 \phi_{0,1}(z; \tau). \quad (2.4)$$

Note that the class  $1A$  consists of the identity element and hence  $Z_{1A}$  is the original untwisted elliptic genus.

In the case of classes  $pA$  ( $p = 2, 3, 5, 7$ ) there is a general formula for the twisted elliptic genera [2]

$$Z_{pA}(z; \tau) = \frac{2}{p+1} \phi_{0,1}(\tau) + \frac{2p}{p+1} \phi_2^{(p)}(\tau) \phi_{-2,1}(z; \tau). \quad (2.5)$$

Explicitly we have

$$Z_{2A}(z; \tau) = \frac{2}{3} \phi_{0,1}(z; \tau) + \frac{4}{3} \phi_2^{(2)}(\tau) \phi_{-2,1}(z; \tau), \quad (2.6)$$

$$Z_{3A}(z; \tau) = \frac{1}{2} \phi_{0,1}(z; \tau) + \frac{3}{2} \phi_2^{(3)}(\tau) \phi_{-2,1}(z; \tau), \quad (2.7)$$

$$Z_{5A}(z; \tau) = \frac{1}{3} \phi_{0,1}(z; \tau) + \frac{5}{3} \phi_2^{(5)}(\tau) \phi_{-2,1}(z; \tau), \quad (2.8)$$

$$Z_{7A}(z; \tau) = \frac{1}{4} \phi_{0,1}(z; \tau) + \frac{7}{4} \phi_2^{(7)}(\tau) \phi_{-2,1}(z; \tau). \quad (2.9)$$

Twisted elliptic genera for other classes are given by [2, 13]

$$Z_{4B}(z; \tau) = \frac{1}{3} \phi_{0,1}(z; \tau) + \left( -\frac{1}{3} \phi_2^{(2)}(\tau) + 2 \phi_2^{(4)}(\tau) \right) \phi_{-2,1}(z; \tau), \quad (2.10)$$

$$Z_{6A}(z; \tau) = \frac{1}{6} \phi_{0,1}(z; \tau) + \left( -\frac{1}{6} \phi_2^{(2)}(\tau) - \frac{1}{2} \phi_2^{(3)}(\tau) + \frac{5}{2} \phi_2^{(6)}(\tau) \right) \phi_{-2,1}(z; \tau), \quad (2.11)$$

$$Z_{8A}(z; \tau) = \frac{1}{6} \phi_{0,1}(z; \tau) + \left( -\frac{1}{2} \phi_2^{(4)}(\tau) + \frac{7}{3} \phi_2^{(8)}(\tau) \right) \phi_{-2,1}(z; \tau), \quad (2.12)$$

$$Z_{11A}(z; \tau) = \frac{1}{6} \phi_{0,1}(z; \tau) + \left( \frac{11}{6} \phi_2^{(11)}(\tau) - \frac{22}{5} [\eta(\tau) \eta(11\tau)]^2 \right) \phi_{-2,1}(z; \tau), \quad (2.13)$$

$$Z_{14A}(z; \tau) = \frac{1}{12} \phi_{0,1}(z; \tau) + \left( -\frac{1}{36} \phi_2^{(2)}(\tau) - \frac{7}{12} \phi_2^{(7)}(\tau) + \frac{91}{36} \phi_2^{(14)}(\tau) - \frac{14}{3} \eta(\tau) \eta(2\tau) \eta(7\tau) \eta(14\tau) \right) \phi_{-2,1}(z; \tau), \quad (2.14)$$

$$Z_{15A}(z; \tau) = \frac{1}{12} \phi_{0,1}(z; \tau) + \left( -\frac{1}{16} \phi_2^{(3)}(\tau) - \frac{5}{24} \phi_2^{(5)}(\tau) + \frac{35}{16} \phi_2^{(15)}(\tau) - \frac{15}{4} \eta(\tau) \eta(3\tau) \eta(5\tau) \eta(15\tau) \right) \phi_{-2,1}(z; \tau), \quad (2.15)$$

$$Z_{23A}(z; \tau) = \frac{1}{12} \phi_{0,1}(z; \tau) + \left( \frac{23}{12} \phi_2^{(23)}(\tau) - \frac{23}{22} f_{23,1}(\tau) - \frac{161}{22} f_{23,2}(\tau) \right) \phi_{-2,1}(z; \tau). \quad (2.16)$$

In the class 23A we have used the newforms [23]

$$f_{23,1}(\tau) = 2q - q^2 - q^4 - 2q^5 - 5q^6 + 2q^7 + 4q^9 + 6q^{10} - 6q^{11} + 5q^{12} + 6q^{13} \\ + 4q^{14} - 10q^{15} - 3q^{16} + 6q^{17} - 2q^{18} - 4q^{19} + \dots,$$

$$f_{23,2}(\tau) = q^2 - 2q^3 - q^4 + 2q^5 + q^6 + 2q^7 - 2q^8 - 2q^{10} - 2q^{11} + q^{12} \\ + 2q^{15} + 3q^{16} - 2q^{17} + 2q^{18} + \dots.$$

### 3. TWISTED ELLIPTIC GENUS OF THE SET TYPE II

Our main task in this paper is to obtain all the twisted elliptic genera belonging to type II. Here, unfortunately there is no definite guiding principle. We have to make an educated guess for the candidate elliptic genera which reproduce the correct coefficients of lower order  $q$ -expansions (see Table 3) and have the correct weight and level as modular forms.

By trial and error we have obtained the following elliptic genera which are written in the form of  $\eta$ -product;

$$Z_{2B}(z; \tau) = 2 \frac{\eta(\tau)^8}{\eta(2\tau)^4} \phi_{-2,1}(z; \tau), \quad (3.1)$$

$$Z_{4A}(z; \tau) = 2 \frac{\eta(2\tau)^8}{\eta(4\tau)^4} \phi_{-2,1}(z; \tau), \quad (3.2)$$

$$Z_{4C}(z; \tau) = 2 \frac{\eta(\tau)^4 \eta(2\tau)^2}{\eta(4\tau)^2} \phi_{-2,1}(z; \tau), \quad (3.3)$$

$$Z_{3B}(z; \tau) = 2 \frac{\eta(\tau)^6}{\eta(3\tau)^2} \phi_{-2,1}(z; \tau), \quad (3.4)$$

$$Z_{6B}(z; \tau) = 2 \frac{\eta(\tau)^2 \eta(2\tau)^2 \eta(3\tau)^2}{\eta(6\tau)^2} \phi_{-2,1}(z; \tau), \quad (3.5)$$

$$Z_{12B}(z; \tau) = 2 \frac{\eta(\tau)^4 \eta(4\tau) \eta(6\tau)}{\eta(2\tau) \eta(12\tau)} \phi_{-2,1}(z; \tau), \quad (3.6)$$

$$Z_{10A}(z; \tau) = 2 \frac{\eta(\tau)^3 \eta(2\tau) \eta(5\tau)}{\eta(10\tau)} \phi_{-2,1}(z; \tau), \quad (3.7)$$

$$Z_{12A}(z; \tau) = 2 \frac{\eta(\tau)^3 \eta(4\tau)^2 \eta(6\tau)^3}{\eta(2\tau) \eta(3\tau) \eta(12\tau)^2} \phi_{-2,1}(z; \tau). \quad (3.8)$$

For the class 21A one has a linear combination of  $\eta$ -products

$$Z_{21A}(z; \tau) = \left( \frac{7}{3} \frac{\eta(\tau)^3 \eta(7\tau)^3}{\eta(3\tau) \eta(21\tau)} - \frac{1}{3} \frac{\eta(\tau)^6}{\eta(3\tau)^2} \right) \phi_{-2,1}(z; \tau). \quad (3.9)$$

One sees that  $\Sigma_g(\tau)$  is the  $\eta$ -product which is modular on congruence subgroup  $\Gamma_0(\text{ord}(g))$  with character.

We note the following relation among the genera of type I and type II

$$Z_{2A}(z; \tau) + Z_{2B}(z; \tau) = 2 Z_{4B}(z; \tau), \quad (3.10)$$

$$Z_{4A}(z; \tau) + Z_{4B}(z; \tau) = 2 Z_{8A}(z; \tau). \quad (3.11)$$

#### 4. MATHIEU MOONSHINE

In Table 2 the character formula for the Mathieu group  $M_{24}$  is presented. We denote its elements as  $\chi_R^g$  where  $R$  runs over irreducible representations and  $g$  runs over conjugacy classes. It is well-known that the character formula obeys the orthogonality relation

$$\sum_g n_g \chi_{R'}^g \bar{\chi}_R^g = |G| \delta_{RR'} \quad (4.1)$$

where  $n_g$  is the number of elements in the conjugacy class  $g$  and  $|G|$  is the order of the group  $G$ . Let us denote the multiplicity of the representation  $R$  in the decomposition of the  $K3$  elliptic genus at level  $n$  as  $c_R(n)$ . We then obtain the value of the twisted genus of the class  $g$  at level  $n$  as

$$\sum_R c_R(n) \chi_R^g = A_g(n), \quad (4.2)$$

where  $A_g(n)$  is defined in (1.10). Note that by choosing  $g = 1A$  in (4.2) we find

$$\sum_R c_n(R) \chi_R^{1A} = \sum_R c_n(R) \dim R = A(n) \quad (4.3)$$

In fact  $c_R(n)$  is the multiplicity of representation  $R$  at level  $n$ .

If one uses the orthogonality relation (4.1), we can invert the relation (4.2) and find a formula for the multiplicities

$$\sum_g \frac{1}{|G|} n_g \bar{\chi}_R^g A_g(n) = c_R(n). \quad (4.4)$$

We have checked by computer that the multiplicities  $c_R(n)$  are positive integers for all representations up to  $n = 600$ . See Table 4. This provides a very strong support of the Mathieu moonshine conjecture.

#### 5. ENTROPY

In Table 3, we have tabulated the values of  $A_g(n)$ , the expansion coefficients of twisted genera  $Z_g(z; \tau)$ . In the untwisted case ( $g = 1A$ ) we have applied the method of Bringmann–Ono [1] and obtained the Poincaré series [6]

$$A(n) = \frac{-2\pi i}{(8n-1)^{\frac{1}{4}}} \sum_{c=1}^{\infty} \frac{1}{\sqrt{c}} I_{\frac{1}{2}} \left( \frac{\pi \sqrt{8n-1}}{2c} \right) \sum_{\substack{k \bmod 4c \\ k^2 \equiv -8n+1 \bmod 8c}} \left( \frac{-4}{k} \right) e^{\frac{k}{2c}\pi i}, \quad (5.1)$$

where  $\left( \frac{-4}{\bullet} \right)$  is the Legendre symbol, and  $I$  denotes the Bessel function,

$$I_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sinh(x).$$

We have identified the exponential growth of  $\{A(n)\}$  at large  $n$  as the entropy of  $K3$  surface

$$S_{K3} = \log A(n) \sim 2\pi \sqrt{\frac{1}{2} \left( n - \frac{1}{8} \right)}. \quad (5.2)$$

See [7, 8] for the discussion of entropy of higher-dimensional complex manifolds with reduced holonomy.

In case of  $g \in$  type I, twisted elliptic genus  $Z_g(z; \tau)$  is modular on the congruence subgroup  $\Gamma_0(\text{ord}(g))$  of  $SL(2; \mathbb{Z})$ . Correspondingly the  $q$ -series  $\Sigma_g(\tau)$  is a mock theta function on  $\Gamma_0(\text{ord}(g))$ , and by using the same method as above the Fourier coefficients  $A_g(n)$  are given by

$$A_g(n) = \frac{-2\pi i}{(8n-1)^{\frac{1}{4}}} \sum_{\substack{c=1 \\ \text{ord}(g)|c}}^{\infty} \frac{1}{\sqrt{c}} I_{\frac{1}{2}} \left( \frac{\pi \sqrt{8n-1}}{2c} \right) \sum_{\substack{k \pmod{4c} \\ k^2 \equiv -8n+1 \pmod{8c}}} \left( \frac{-4}{k} \right) e^{\frac{k}{2c}\pi i}. \quad (5.3)$$

See [6] where a case of  $\Gamma_0(2)$  was studied. The above formula shows that the entropy  $S_g$  of “twisted”  $K3$  is given by

$$S_g = \log |A_g(n)| \sim \frac{S_{K3}}{\text{ord}(g)}. \quad (5.4)$$

Thus the entropy of twisted  $K3$  is reduced by a factor  $1/\text{ord}(g)$ . This coincides with the result of [20] that the entropy of the  $\mathbb{Z}_N$  twisted CHL model is  $1/N$  times the entropy of the untwisted model.

## 6. DISCUSSIONS

We have completed the analysis initiated in [2, 13] on Mathieu moonshine phenomenon by providing all the twisted elliptic genera for  $K3$  surface. Making use of them we are able to decompose uniquely the expansion coefficients  $\{A_g(n)\}$  into a sum of irreducible representations of  $M_{24}$ . We find that the multiplicities of all irreducible representations are positive integers up to the level  $n = 600$ .

For the sake of illustration we present the decomposition at the level  $n = 98$ ;

$$\begin{aligned} & 1754939889054075390 \\ &= 7168167560 \times 1 + 164868700882 \times 23 + 1806385660318 \times 252 \\ &+ 1813554671156 \times 253 + 12694876811718 \times 1771 + 25232056046588 \times 3520 \\ &+ 322568604932 \times (45 + \overline{45}) + 7096515632052 \times (990 + \overline{990}) \\ &+ 7419084236984 \times (1035 + \overline{1035}) + 7419083183322 \times 1035' \\ &+ 1655854140602 \times (231 + \overline{231}) + 5519511336942 \times (770 + \overline{770}) \\ &+ 3462239800920 \times 483 + 9067771260936 \times 1265 + 14508430647818 \times 2024 \\ &+ 16321986797048 \times 2277 + 23741069980370 \times 3312 + 38084633405380 \times 5313 \\ &+ 41546870254732 \times 5796 + 39740484586192 \times 5544 + 74513414138524 \times 10395 \end{aligned}$$

Thus the observation of [9] may well be proved to be true.

We are, however, still very far from satisfactory understanding of the origin of the symmetry of the Mathieu group  $M_{24}$ . As is well-known, there are special classes of  $K3$  surfaces which possess automorphism under subgroups of  $M_{23}$  [18, 19] (see [22] for a recent result). Thus it appears that  $M_{24}$  emerges as an enhanced symmetry in string

theory. Hopefully the twisted genera we have obtained offer some clue in our search for the action of  $M_{24}$  on the string Hilbert space in  $K3$  compactification.

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#### APPENDIX A. JACOBI THETA FUNCTIONS

The Jacobi theta functions are defined by

$$\begin{aligned}
 \theta_{11}(z; \tau) &= \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}(n+\frac{1}{2})^2} e^{2\pi i(n+\frac{1}{2})(z+\frac{1}{2})}, \\
 \theta_{10}(z; \tau) &= \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}(n+\frac{1}{2})^2} e^{2\pi i(n+\frac{1}{2})z}, \\
 \theta_{00}(z; \tau) &= \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}n^2} e^{2\pi in z}, \\
 \theta_{01}(z; \tau) &= \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}n^2} e^{2\pi in(z+\frac{1}{2})}.
 \end{aligned} \tag{A.1}$$

Throughout this paper, we set  $q = e^{2\pi i\tau}$  with  $\tau$  in the upper half plane,  $\tau \in \mathbb{H}$ .



conjugacy class	cycle shape	
1A	$1^{24}$	()
2A	$1^8 \cdot 2^8$	(1, 8)(2, 12)(4, 15)(5, 7)(9, 22)(11, 18)(14, 19)(23, 24)
3A	$1^6 \cdot 3^6$	(3, 18, 20)(4, 22, 24)(5, 19, 17)(6, 11, 8)(7, 15, 10)(9, 12, 14)
5A	$1^4 \cdot 5^4$	(2, 21, 13, 16, 23)(3, 5, 15, 22, 14)(4, 12, 20, 17, 7)(9, 18, 19, 10, 24)
4B	$1^4 \cdot 2^2 \cdot 4^4$	(1, 17, 21, 9)(2, 13, 24, 15)(3, 23)(4, 14, 5, 8)(6, 16)(12, 18, 20, 22)
7A	$1^3 \cdot 7^3$	(1, 17, 5, 21, 24, 10, 6)(2, 12, 13, 9, 4, 23, 20)(3, 8, 22, 7, 18, 14, 19)
7B	$1^3 \cdot 7^3$	(1, 21, 6, 5, 10, 17, 24)(2, 9, 20, 13, 23, 12, 4)(3, 7, 19, 22, 14, 8, 18)
8A	$1^2 \cdot 2^1 \cdot 4^1 \cdot 8^2$	(1, 13, 17, 24, 21, 15, 9, 2)(3, 16, 23, 6)(4, 22, 14, 12, 5, 18, 8, 20)(7, 11)
6A	$1^2 \cdot 2^2 \cdot 3^2 \cdot 6^2$	(1, 8)(2, 24, 11, 12, 23, 18)(3, 20, 10)(4, 15)(5, 19, 9, 7, 14, 22)(6, 16, 13)
11A	$1^2 \cdot 11^2$	(1, 3, 10, 4, 14, 15, 5, 24, 13, 17, 18)(2, 21, 23, 9, 20, 19, 6, 12, 16, 11, 22)
15A	$1^1 \cdot 3^1 \cdot 5^1 \cdot 15^1$	(2, 13, 23, 21, 16)(3, 7, 9, 5, 4, 18, 15, 12, 19, 22, 20, 10, 14, 17, 24)(6, 8, 11)
15B	$1^1 \cdot 3^1 \cdot 5^1 \cdot 15^1$	(2, 23, 16, 13, 21)(3, 12, 24, 15, 17, 18, 14, 4, 10, 5, 20, 9, 22, 7, 19)(6, 8, 11)
14A	$1^1 \cdot 2^1 \cdot 7^1 \cdot 14^1$	(1, 12, 17, 13, 5, 9, 21, 4, 24, 23, 10, 20, 6, 2)(3, 18, 8, 14, 22, 19, 7)(11, 15)
14B	$1^1 \cdot 2^1 \cdot 7^1 \cdot 14^1$	(1, 13, 21, 23, 6, 12, 5, 4, 10, 2, 17, 9, 24, 20)(3, 14, 7, 8, 19, 18, 22)(11, 15)
23A	$1^1 \cdot 23^1$	(1, 7, 6, 24, 14, 4, 16, 12, 20, 9, 11, 5, 15, 10, 19, 18, 23, 17, 3, 2, 8, 22, 21)
23B	$1^1 \cdot 23^1$	(1, 4, 11, 18, 8, 6, 12, 15, 17, 21, 14, 9, 19, 2, 7, 16, 5, 23, 22, 24, 20, 10, 3)
12B	$12^2$	(1, 12, 24, 23, 10, 8, 18, 6, 3, 21, 2, 7)(4, 9, 11, 15, 13, 16, 20, 5, 22, 17, 14, 19)
6B	$6^4$	(1, 24, 10, 18, 3, 2)(4, 11, 13, 20, 22, 14)(5, 17, 19, 9, 15, 16)(6, 21, 7, 12, 23, 8)
4C	$4^6$	(1, 23, 18, 21)(2, 12, 10, 6)(3, 7, 24, 8)(4, 15, 20, 17)(5, 14, 9, 13)(11, 16, 22, 19)
3B	$3^8$	(1, 10, 3)(2, 24, 18)(4, 13, 22)(5, 19, 15)(6, 7, 23)(8, 21, 12)(9, 16, 17)(11, 20, 14)
2B	$2^{12}$	(1, 8)(2, 10)(3, 20)(4, 22)(5, 17)(6, 11)(7, 15)(9, 13)(12, 14)(16, 18)(19, 23)(21, 24)
10A	$2^2 \cdot 10^2$	(1, 8)(2, 18, 21, 19, 13, 10, 16, 24, 23, 9)(3, 4, 5, 12, 15, 20, 22, 17, 14, 7)(6, 11)
21A	$3^1 \cdot 21^1$	(1, 3, 9, 15, 5, 12, 2, 13, 20, 23, 17, 4, 14, 10, 21, 22, 19, 6, 7, 11, 16)(8, 18, 24)
21B	$3^1 \cdot 21^1$	(1, 12, 17, 22, 16, 5, 23, 21, 11, 15, 20, 10, 7, 9, 13, 14, 6, 3, 2, 4, 19)(8, 24, 18)
4A	$2^4 \cdot 4^4$	(1, 4, 8, 15)(2, 9, 12, 22)(3, 6)(5, 24, 7, 23)(10, 13)(11, 14, 18, 19)(16, 20)(17, 21)
12A	$2^1 \cdot 4^1 \cdot 6^1 \cdot 12^1$	(1, 15, 8, 4)(2, 19, 24, 9, 11, 7, 12, 14, 23, 22, 18, 5)(3, 13, 20, 6, 10, 16)(17, 21)

TABLE 1. Representatives of conjugacy classes.

TABLE 2. Character table of the Mathieu group  $M_{24}$ . Here we have used  $e_p^\pm = \frac{1}{2} (\pm \sqrt{-p} - 1)$ .

1A	2A	3A	5A	4B	7A	7B	8A	6A	11A	15A	15B	14A	14B	23A	23B	12B	6B	4C	3B	2B	10A	21A	21B	4A	12A
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
23	7	5	3	3	2	2	1	1	1	0	0	0	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
252	28	9	2	4	0	0	0	1	-1	-1	-1	0	0	-1	-1	0	0	0	0	12	2	0	0	4	1
253	13	10	3	1	1	1	-1	-2	0	0	0	-1	-1	0	0	1	1	1	1	-11	-1	1	1	-3	0
1771	-21	16	1	-5	0	0	-1	0	0	1	1	0	0	0	0	-1	-1	-1	7	11	1	0	0	3	0
3520	64	10	0	0	-1	-1	0	-2	0	0	0	1	1	1	1	0	0	0	-8	0	0	-1	-1	0	0
45	-3	0	0	1	$e_7^+$	$e_7^-$	-1	0	1	0	0	$-e_7^+$	$-e_7^-$	-1	-1	1	-1	1	3	5	0	$e_7^-$	$e_7^+$	-3	0
$\overline{45}$	-3	0	0	1	$e_7^-$	$e_7^+$	-1	0	1	0	0	$-e_7^-$	$-e_7^+$	-1	-1	1	-1	1	3	5	0	$e_7^+$	$e_7^-$	-3	0
990	-18	0	0	2	$e_7^+$	$e_7^-$	0	0	0	0	0	$e_7^+$	$e_7^-$	1	1	1	-1	-2	3	-10	0	$e_7^-$	$e_7^+$	6	0
$\overline{990}$	-18	0	0	2	$e_7^-$	$e_7^+$	0	0	0	0	0	$e_7^-$	$e_7^+$	1	1	1	-1	-2	3	-10	0	$e_7^+$	$e_7^-$	6	0
1035	-21	0	0	3	$2e_7^+$	$2e_7^-$	-1	0	1	0	0	0	0	0	0	-1	1	-1	-3	-5	0	$-e_7^-$	$-e_7^+$	3	0
$\overline{1035}$	-21	0	0	3	$2e_7^-$	$2e_7^+$	-1	0	1	0	0	0	0	0	0	-1	1	-1	-3	-5	0	$-e_7^+$	$-e_7^-$	3	0
1035'	27	0	0	-1	-1	-1	1	0	1	0	0	-1	-1	0	0	0	2	3	6	35	0	-1	-1	3	0
231	7	-3	1	-1	0	0	-1	1	0	$e_{15}^+$	$e_{15}^-$	0	0	1	1	0	0	3	0	-9	1	0	0	-1	-1
$\overline{231}$	7	-3	1	-1	0	0	-1	1	0	$e_{15}^-$	$e_{15}^+$	0	0	1	1	0	0	3	0	-9	1	0	0	-1	-1
770	-14	5	0	-2	0	0	0	1	0	0	0	0	0	$e_{23}^+$	$e_{23}^-$	1	1	-2	-7	10	0	0	0	2	-1
$\overline{770}$	-14	5	0	-2	0	0	0	1	0	0	0	0	0	$e_{23}^-$	$e_{23}^+$	1	1	-2	-7	10	0	0	0	2	-1
483	35	6	-2	3	0	0	-1	2	-1	1	1	0	0	0	0	0	0	3	0	3	-2	0	0	3	0
1265	49	5	0	1	-2	-2	1	1	0	0	0	0	0	0	0	0	0	-3	8	-15	0	1	1	-7	-1
2024	8	-1	-1	0	1	1	0	-1	0	-1	-1	1	1	0	0	0	0	0	8	24	-1	1	1	8	-1
2277	21	0	-3	1	2	2	-1	0	0	0	0	0	0	0	0	0	2	-3	6	-19	1	-1	-1	-3	0
3312	48	0	-3	0	1	1	0	0	1	0	0	-1	-1	0	0	0	-2	0	-6	16	1	1	1	0	0
5313	49	-15	3	-3	0	0	-1	1	0	0	0	0	0	0	0	0	0	-3	0	9	-1	0	0	1	1
5796	-28	-9	1	4	0	0	0	-1	-1	1	1	0	0	0	0	0	0	0	0	36	1	0	0	-4	-1
5544	-56	9	-1	0	0	0	0	1	0	-1	-1	0	0	1	1	0	0	0	0	24	-1	0	0	-8	1
10395	-21	0	0	-1	0	0	1	0	0	0	0	0	0	-1	-1	0	0	3	0	-45	0	0	0	3	0

TABLE 3. Values  $A_g(n)$  from twisted elliptic genera for lower levels  $n$ .

$n$	1A	2A	3A	5A	4B	7A	8A	6A	11A	15A	14A	23A	12B	6B	4C	3B	2B	10A	21A	4A	12A
1	90	-6	0	0	2	-1	-2	0	2	0	1	-2	2	-2	2	6	10	0	-1	-6	0
2	462	14	-6	2	-2	0	-2	2	0	-1	0	2	0	0	6	0	-18	2	0	-2	-2
3	1540	-28	10	0	-4	0	0	2	0	0	0	-1	2	2	-4	-14	20	0	0	4	-2
4	4554	42	0	-6	2	4	-2	0	0	0	0	0	0	4	-6	12	-38	2	-2	-6	0
5	11592	-56	-18	2	8	0	0	-2	-2	2	0	0	0	0	0	0	72	2	0	-8	-2
6	27830	86	20	0	-2	-2	2	-4	0	0	2	0	0	0	6	-16	-90	0	-2	6	0
7	61686	-138	0	6	-10	2	-2	0	-2	0	2	0	-2	-2	-2	30	118	-2	2	6	0
8	131100	188	-30	0	4	-3	0	2	2	0	-1	0	0	0	-12	0	-180	0	0	-4	2
9	265650	-238	42	-10	10	0	-2	2	0	2	0	0	-2	6	10	-42	258	-2	0	-14	-2
10	521136	336	0	6	-8	0	-4	0	0	0	0	2	-2	2	16	42	-352	-2	0	0	0
11	988770	-478	-60	0	-14	6	2	-4	2	0	-2	0	0	0	-6	0	450	0	0	18	0
12	1830248	616	62	8	8	0	0	-2	2	2	0	0	2	-6	-16	-70	-600	0	0	-8	-2
13	3303630	-786	0	0	22	-6	2	0	0	0	-2	2	0	-4	6	84	830	0	0	-18	0
14	5844762	1050	-90	-18	-6	0	2	6	0	0	0	2	0	0	18	0	-1062	-2	0	10	-2
15	10139734	-1386	118	4	-26	-4	-2	6	0	-2	0	0	2	2	-10	-110	1334	4	2	22	-2
16	17301060	1764	0	0	12	0	0	0	-4	0	0	0	2	6	-28	126	-1740	0	0	-12	0
17	29051484	-2212	-156	14	28	0	-4	-4	0	-1	0	0	0	0	12	0	2268	-2	0	-36	0
18	48106430	2814	170	0	-18	8	-2	-6	-2	0	0	-2	2	-6	38	-166	-2850	0	2	14	2
19	78599556	-3612	0	-24	-36	0	0	0	2	0	0	0	-2	-6	-20	210	3540	0	0	36	0
20	126894174	4510	-228	14	14	-6	-2	4	0	2	2	0	0	0	-42	0	-4482	-2	0	-18	0
21	202537080	-5544	270	0	48	4	4	6	-2	0	0	0	-2	6	16	-282	5640	0	-2	-40	2
22	319927608	6936	0	18	-16	-7	4	0	0	0	-1	0	0	4	48	300	-6968	2	-1	24	0
23	500376870	-8666	-360	0	-58	0	-2	-8	4	0	0	2	0	0	-18	0	8550	0	0	54	0
24	775492564	10612	400	-36	28	0	0	-8	0	0	0	0	0	-8	-60	-392	-10556	4	0	-28	-4
25	1191453912	-12936	0	12	64	12	-4	0	0	0	0	0	2	-10	32	462	13064	4	0	-72	0
26	1815754710	15862	-510	0	-34	0	-6	10	0	0	0	-1	0	0	78	0	-15930	0	0	22	-2
27	2745870180	-19420	600	30	-76	-10	4	8	-2	0	-2	0	0	8	-36	-600	19268	-2	2	84	0
28	4122417420	23532	0	0	36	2	0	0	0	0	-2	0	0	12	-84	660	-23460	0	2	-36	0
29	6146311620	-28348	-762	-50	100	-6	4	-10	-2	-2	2	0	0	0	36	0	28548	-2	0	-92	-2
30	9104078592	34272	828	22	-40	0	4	-12	4	-2	0	0	0	-8	96	-840	-34352	-2	0	48	0
31	13401053820	-41412	0	0	-116	0	-4	0	0	0	0	-2	-2	-10	-44	966	41180	0	0	108	0
32	19609321554	49618	-1062	34	50	18	2	10	-2	-2	2	0	0	0	-126	0	-49518	2	0	-46	2
33	28530824630	-59178	1220	0	126	0	-6	12	0	0	0	2	-4	12	62	-1204	59430	0	0	-138	0
34	41286761478	70758	0	-72	-66	-10	-6	0	6	0	2	0	0	12	150	1332	-70890	0	2	54	0
35	59435554926	-84530	-1518	26	-154	6	2	-14	0	2	2	0	0	0	-66	0	84222	2	0	158	2
36	85137361430	100310	1670	0	70	-12	-2	-10	0	0	0	0	-2	-18	-170	-1666	-100170	0	0	-74	-2

TABLE 4. Multiplicities  $c_R(n)$  in decomposition of  $A_g(n)$ .

$n$	1	23	252	253	1771	3520	$\frac{45}{45}$	$\frac{990}{990}$	$\frac{1035}{1035}$	1035'	$\frac{231}{231}$	$\frac{770}{770}$	483	265	2024	2277	3312	5313	5796	5544	10395
1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	0	0
6	0	0	0	0	0	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2
7	0	0	0	0	2	0	0	0	0	0	0	0	0	0	2	0	0	2	2	2	2
8	0	0	0	0	0	2	0	1	1	0	0	0	0	2	0	2	2	4	2	2	6
9	0	0	0	0	2	4	0	0	2	2	0	2	2	0	2	2	4	4	8	8	10
10	0	0	0	2	4	8	0	2	2	2	2	0	2	4	4	6	6	12	10	10	24
11	0	0	0	0	8	12	0	4	4	6	0	4	0	2	10	8	14	22	26	24	40
12	0	2	2	4	12	30	0	8	8	4	2	6	4	12	12	18	26	40	40	38	80
13	0	0	4	2	26	44	2	14	14	18	2	10	6	16	30	28	44	70	84	80	136
14	0	0	4	6	38	86	0	24	24	22	8	16	14	34	46	58	80	128	132	126	254
15	0	0	12	8	78	144	2	40	44	46	8	38	18	46	86	88	138	218	246	238	424
16	0	2	18	22	122	252	2	72	72	68	18	50	36	100	140	170	232	378	400	382	742
17	0	2	30	26	212	410	8	116	124	130	25	94	54	140	246	262	392	630	704	670	1222
18	0	6	50	58	342	704	6	194	202	192	50	148	100	256	388	454	654	1044	1120	1074	2058
19	0	4	80	72	582	1116	18	318	332	346	68	252	150	394	664	722	1062	1702	1880	1800	3320
20	0	14	128	138	904	1836	20	516	536	520	126	390	254	676	1036	1196	1716	2764	2980	2846	5408
21	2	20	214	200	1476	2902	40	814	860	872	182	652	396	1020	1684	1862	2742	4384	4828	4622	8572
22	2	32	328	346	2302	4616	55	1298	1348	1336	314	988	640	1686	2630	3000	4324	6950	7532	7204	13620
23	2	40	512	496	3638	7166	98	2020	2118	2144	460	1590	972	2546	4162	4624	6768	10856	11898	11376	21204
24	0	80	798	824	5584	11192	132	3140	3278	3236	744	2426	1544	4050	6376	7248	10500	16834	18294	17504	32976
25	8	108	1232	1208	8654	17084	234	4814	5038	5084	1106	3764	2336	6108	9892	11042	16112	25840	28288	27056	50524
26	6	174	1860	1904	13090	26148	322	7348	7670	7626	1742	5677	3602	9444	14968	16940	24566	39428	42894	41022	77176
27	12	252	2836	2802	19914	39436	514	11092	11618	11666	2560	8688	5394	14100	22744	25462	37148	59564	65114	62294	116494
28	16	398	4238	4310	29772	59330	742	16686	17418	17356	3922	12912	8160	21414	34026	38434	55764	89490	97456	93218	175146
29	26	560	6328	6286	44512	88280	1154	24840	25994	26078	5758	19380	12090	31636	50892	57068	83146	133356	145690	139342	260828
30	34	876	9368	9486	65776	131020	1642	36824	38480	38368	8642	28580	18008	47172	75158	84776	123176	197596	215318	205970	386724
31	58	1236	13802	13764	97060	192538	2500	54178	56660	56800	12582	42218	26384	69082	110920	124506	181274	290780	317502	303700	568798
32	76	1866	20166	20356	141714	282074	3564	79320	82884	82730	18576	61574	38738	101530	161978	182554	265284	425624	463950	443760	832834
33	122	2664	29396	29374	206524	410062	5286	115334	120644	120798	26830	89868	56226	147156	236010	265136	385974	619072	675796	646432	1211106
34	166	3900	42474	42810	298508	593800	7542	166990	174510	174330	39066	129694	81546	213644	341154	384250	558530	896052	977004	934530	1753356
35	248	5536	61184	61234	430134	854284	10988	240304	251292	251544	55956	187094	117138	306736	491602	552494	804038	1289768	1407604	1346380	2523178
36	334	8058	87622	88196	615626	1224424	15560	344314	359902	359564	80470	267604	168092	440318	703542	792158	1151786	1847690	2014952	1927370	3615350

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YUKAWA INSTITUTE FOR THEORETICAL PHYSICS, KYOTO UNIVERSITY, KYOTO 606–8502, JAPAN

*E-mail address:* [eguchi@yukawa.kyoto-u.ac.jp](mailto:eguchi@yukawa.kyoto-u.ac.jp)

DEPARTMENT OF MATHEMATICS, NARUTO UNIVERSITY OF EDUCATION, TOKUSHIMA 772-8502, JAPAN.

*E-mail address:* [KHikami@gmail.com](mailto:KHikami@gmail.com)